MATH 1010E University Mathematics
Lecture Notes (week 10)
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1 A first look at the Fundamental Theorem of Calculus

Last week, we have defined the definite integrals of a function f defined on a closed and bounded interval $[a, b]$. If the function f is continuous, we know that the definite integral $\int_a^b f(x)\, dx$ exists and can be computed by the method of *Riemann sum*:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

We have computed a few simple examples explicitly. From these examples, we see that the Riemann sum approach requires some infinite sum, which becomes very hard to calculate unless for very simples functions. Hence, we need some other ways to do calculations of definite integrals.

Recall that we have introduced another kind of integration called *indefinite integral*, which finds the primitive function $F(x)$ of a given function $f(x)$, i.e. $F'(x) = f(x)$. In fact, the two kinds of integrals: *indefinite integrals* and *definite integrals* are closely related. If we know the indefinite integral $F(x)$ of $f(x)$ then we can compute the definite integral of $f(x)$ very easily without going through all those Riemann sum constructions.

To be more precise, if $F : [a, b] \to \mathbb{R}$ is a function which is continuous on $[a.b]$ and differentiable in (a, b) such that $F'(x) = f'(x)$ on (a, b), then

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{1.1}$$

In other words, the definite integral of $f(x)$ is just the difference of the value of its primitive function $F(x)$ at the end points.

We will give a more precise statement of the Fundamental Theorem of Calculus next week. Let us first make two remarks about it. First of all, remember that the primitive function $F(x)$ is determined by $f(x)$ only up an integration constant. For example, both $F(x) = x$ and $F(x) = x + 1$ satisfies $F'(x) = 1$. In (1) above, we can use either one of them because the integration constant would cancel itself out in the difference on the right hand side anyway:

$$\int_0^1 1\,dx = [x]_0^1 = 1 - 0 = 1 = 2 - 1 = [x+1]_0^1 = \int_0^1 1\,dx.$$

Second, we can rewrite (1) as

$$\int_a^b F'(x)\,dx = F(b) - F(a),$$

which says that differentiating first and then integrating just returns the original function (evaluated at the end points and take the difference). This is telling us that differentiation and integration are inverse process of each other.

Let us now make use of (1) to compute some more complicated examples.

Example 1.1 *A primitive function of $f(x) = x$ is given by $F(x) = \frac{x^2}{2}$. Therefore,*

$$\int_0^1 x\,dx = F(1) - F(0) = \frac{1}{2} - 0 = \frac{1}{2}.$$

This agrees with the answer we got from a Riemann sum calculation last week.

Example 1.2 *Since we know that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq 1$, therefore*

$$\int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1} - 0 = \frac{1}{n+1}.$$

Question: Can we compute the definite integral above using Riemann sum? What happens for the case $n = -1$?

Example 1.3 *We can use (1) to compute more complicated definite integrals given that we can find the primitive function (i.e. to solve the indefinite integral).*

$$\int_1^3 \frac{3x^3 - 5}{x^2} dx = \int_1^3 \left(3x - \frac{5}{x^2}\right)\,dx = \left[\frac{3x^2}{2} + \frac{5}{x}\right]_1^3 = \frac{26}{3}.$$

Example 1.4 *We can also combine (1) and the u-substitution method. For example, consider the definite integral*

$$\int_0^1 x\sqrt{1 - x^2}\,dx.$$

If we let $u = 1 - x^2$, then $du = -2xdx$. Moreover, we also have to change the "limits of integration". When $x = 0$, we have $u = 1$; when $x = 1$, we have $u = 0$. Therefore, in the new variable u, the definite integral becomes

$$\int_0^1 x\sqrt{1-x^2}\,dx = \int_1^0 -\frac{\sqrt{u}}{2}du = \int_0^1 \frac{\sqrt{u}}{2}du = \left[\frac{u^{3/2}}{3}\right]_0^1 = \frac{1}{3}.$$

Example 1.5 *We have to keep in mind which variable are the limits of integration corresponding to to avoid miscalculations. Consider the definite integral*

$$\int_0^{\pi/4} \sec^2 x \tan x\,dx.$$

If we do the u-substitution without explicitly mentioning the variable u, then we have to substitute the value of x in the end.

$$\int_0^{\pi/4} \sec^2 x \tan x\,dx = \int_0^{\pi/4} \tan x\,d(\tan x) = \left[\frac{\tan^2 x}{2}\right]_{x=0}^{x=\pi/4} = \frac{1}{2} - 0 = \frac{1}{2}.$$

In this example we can also do a different substitution.

$$\int_0^{\pi/4} \sec^2 x \tan x\,dx = \int_0^{\pi/4} \sec x\,d(\sec x) = \left[\frac{\sec^2 x}{2}\right]_{x=0}^{x=\pi/4} = 1 - \frac{1}{2} = \frac{1}{2}.$$